

# NONEXISTENCE OF TIGHT SPHERICAL DESIGN OF HARMONIC INDEX 4

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**ABSTRACT.** We give a new upper bound of the cardinality of a set of equiangular lines in  $\mathbb{R}^n$  with a fixed angle  $\theta$  for each  $(n, \theta)$  satisfying certain conditions. Our techniques are based on semi-definite programming methods for spherical codes introduced by Bachoc–Vallentin [J. Amer. Math. Soc. 2008]. As a corollary to our bound, we show the nonexistence of spherical tight designs of harmonic index 4 on  $S^{n-1}$  with  $n \geq 3$ .

## 1. INTRODUCTION

The purpose of this paper is to give a new upper bound of the cardinality of a set of equiangular lines with certain angles (see Theorem 2.1). As a corollary to our bound, we show the nonexistence of tight designs of harmonic index 4 on  $S^{n-1}$  with  $n \geq 3$  (see Theorem 1.1).

Throughout this paper,  $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$  denotes the unit sphere in  $\mathbb{R}^n$ . By Bannai–Okuda–Tagami [6], a finite subset  $X$  of  $S^{n-1}$  is called a *spherical design of harmonic index  $t$  on  $S^{n-1}$*  (or shortly, a *harmonic index  $t$ -design on  $S^{n-1}$* ) if  $\sum_{\mathbf{x} \in X} f(\mathbf{x}) = 0$  for any harmonic polynomial function  $f$  on  $\mathbb{R}^n$  of degree  $t$ .

Our concern in this paper is in tight harmonic index 4-designs. A harmonic index  $t$ -design  $X$  is said to be *tight* if  $X$  attains the lower bound given by [6, Theorem 1.2]. In particular, for  $t = 4$ , a harmonic index 4-design on  $S^{n-1}$  is tight if its cardinality is  $(n+1)(n+2)/6$ . For the cases where  $n = 2$ , we can construct tight harmonic index 4-designs as two points  $\mathbf{x}$  and  $\mathbf{y}$  on  $S^1$  with the inner-product  $\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^2} = \pm\sqrt{1/2}$ .

The paper [6, Theorem 4.2] showed that if tight harmonic index 4-designs on  $S^{n-1}$  exist, then  $n$  must be 2 or  $3(2k-1)^2 - 4$  for some integers  $k \geq 3$ . As a main result of this paper, we show that the later cases do not occur. That is, the following theorem holds:

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**Theorem 1.1.** *For each  $n \geq 3$ , spherical tight design of harmonic index 4 on  $S^{n-1}$  does not exist.*

A set of lines in  $\mathbb{R}^n$  is called an *equiangular line system* if the angle between each pair of lines is constant. By definition, an equiangular line system can be considered as a spherical two-distance set with the inner product set  $\{\pm \cos \theta\}$  for some constant  $\theta$ . Such the constant  $\theta$  is called *the common angle* of the equiangular line system. The recent development of this topic can be found in [5, 10].

By [6, Proposition 4.2], any tight harmonic index 4-design on  $S^{n-1}$  can be considered as an equiangular line system with the common angle  $\arccos \sqrt{3/(n+4)}$ . The proof of Theorem 1.1 will be reduced to a new relative upper bound (see Theorem 2.1) for the cardinalities of equiangular line systems with a fixed common angle. Note that in some cases, our relative bound is better than the Lemmens–Seidel relative bound (see Section 2 for more details).

The paper is organized as follows: In Section 2, as a main theorem of this paper, we give a new relative bound for the cardinalities of equiangular line systems with a fixed common angle satisfying certain conditions. Theorem 1.1 is followed as a corollary to our relative bound. In Section 3, our relative bound is proved based on the method by Bachoc–Vallentin [1].

## 2. MAIN RESULTS

In this paper, we denote by  $M(n)$  and  $M_{\cos \theta}(n)$  the maximum number of equiangular lines in  $\mathbb{R}^n$  and that with the fixed common angle  $\theta$ , respectively. By definition,

$$M(n) = \sup_{0 \leq \alpha < 1} M_\alpha(n).$$

The important problems for equiangular lines are to give upper and lower estimates  $M(n)$  or  $M_\alpha(n)$  for fixed  $\alpha$ . One can find a summary of recent progress of this topic in [5, 10].

Let us fix  $0 \leq \alpha < 1$ . Then for a finite subset  $X$  of  $S^{n-1}$  with  $I(X) \subset \{\pm \alpha\}$ , we can easily find an equiangular line system with the common angle  $\arccos \alpha$  and the cardinality  $|X|$ , where

$$I(X) := \{\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^n} \mid \mathbf{x}, \mathbf{y} \in X \text{ with } \mathbf{x} \neq \mathbf{y}\}$$

is the set of inner-product values of distinct vectors in  $X \subset S^{n-1} \subset \mathbb{R}^n$ . The converse is also true. In particular, we have

$$M_\alpha(n) = \max\{|X| \mid X \subset S^{n-1} \text{ with } I(X) \subset \{\pm \alpha\}\},$$

and therefore, our problem can be considered as a problem in special kinds of spherical two-distance sets.

In this paper, we are interested in upper estimates of  $M_\alpha(n)$ . According to [11], Gerzon gave the upper bound on  $M(n)$  as  $M(n) \leq n(n+1)/2$  and therefore, we have

$$M_\alpha(n) \leq \frac{n(n+1)}{2} \quad \text{for any } \alpha.$$

This upper bound is called the Gerzon absolute bound. Lemmens and Seidel [11] showed that

$$M_\alpha(n) \leq \frac{n(1-\alpha^2)}{1-n\alpha^2} \quad \text{in the cases where } 1-n\alpha^2 > 0.$$

This inequality is sometimes called the Lemmens–Seidel relative bound as opposed to the Gerzon absolute bound.

As a main theorem of this paper, we give other upper estimates of  $M_\alpha(n)$  as follows:

**Theorem 2.1.** *Let us take  $n \geq 3$  and  $\alpha \in (0, 1)$  with*

$$2 - \frac{6\alpha - 3}{\alpha^2} < n < 2 + \frac{6\alpha + 3}{\alpha^2}.$$

*Then*

$$M_\alpha(n) \leq 2 + \frac{(n-2)}{\alpha} \max \left\{ \frac{(1-\alpha)^3}{(n-2)\alpha^2 + 6\alpha - 3}, \frac{(1+\alpha)^3}{-(n-2)\alpha^2 + 6\alpha + 3} \right\}.$$

*In particular, for an integer  $l \geq 2$ , if*

$$3l^2 - 6l + 2 < n < 3l^2 + 6l + 2$$

*then*

$$M_{1/l}(n) \leq 2 + (n-2) \max \left\{ \frac{(l-1)^3}{-3l^2 + 6l + (n-2)}, \frac{(l+1)^3}{3l^2 + 6l - (n-2)} \right\}.$$

Recall that by [6, Proposition 4.2, Theorem 4.2] for  $n \geq 3$ , if there exists a tight harmonic index 4-design  $X$  on  $S^{n-1}$ , then  $n = 3(2k-1)^2 - 4$  for some  $k \geq 3$  and

$$M_{\sqrt{3/(n+4)}}(n) = (n+1)(n+2)/6.$$

However, as a corollary to Theorem 2.1, we have the following upper bound of  $M_{\sqrt{3/(n+4)}}(n)$  and obtain Theorem 1.1.

**Corollary 2.2.** *Let us put  $n_k := 3(2k-1)^2 - 4$  and  $\alpha_k := \sqrt{3/(n_k+4)} = 1/(2k-1)$ . Then for each integer  $k \geq 2$ ,*

$$M_{\alpha_k}(n_k) \leq 2(k-1)(4k^3 - k - 1) < (n_k+1)(n_k+2)/6.$$

It should be remarked that in the setting of Corollary 2.2, the Lemmens–Seidel relative bound does not work since

$$1 - n_k \alpha_k^2 = -2(4k^2 - 4k - 1)/(2k - 1)^2 < 0$$

and our bound is better than the Gerzon absolute bound.

The proof of Theorem 2.1 is given in Section 3 based on Bachoc–Vallentin’s SDP method for spherical codes [1]. The origins of applications of the linear programming method in coding theory can be traced back to the work of Delsarte [8]. Applications of semidefinite programming (SDP) method in coding theory and distance geometry gained momentum after the pioneering work of Schrijver [13] that derived SDP bounds on codes in the Hamming and Johnson spaces. Schrijver’s approach was based on the so-called Terwilliger algebra of the association scheme. The similar idea for spherical codes were formulated by Bachoc and Vallentin [1] regarding for kissing number problems. Barg and Yu [4] modified it to achieve maximum size of spherical two-distance sets in  $\mathbb{R}^n$  for most values of  $n \leq 93$ . In our proof, we restricted the method to obtain upper bounds for equiangular line sets.

We can see in [9, 13] and [1, 2, 3, 12] that the SDP method works well for studying binary codes and spherical codes, respectively. Especially, for equiangular lines, Barg and Yu [5] give the best known upper bounds of  $M(n)$  for some  $n$  with  $n \leq 136$  by the SDP method. Our bounds in Corollary 2.2 are the same as [5] in lower dimensions. However, in some cases, we need some softwares to complete the SDP method. It should be emphasized that our theorem offer upper bound of  $M_{\alpha_k}(n_k)$  for arbitrary large  $n_k$  and the proof can be followed by hand calculations without using any convex optimization software.

### 3. PROOF OF OUR RELATIVE BOUND

To prove Theorem 2.1, we apply Bachoc–Vallentin’s SDP method for spherical codes in [1] to spherical two-distance sets. The explicit statement of it was given by Barg–Yu [4].

We use symbols  $P_l^n(u)$  and  $S_l^n(u, v, t)$  in the sense of [1]. It should be noted that the definition of  $S_l^n(u, v, t)$  is different from that of [2] and [4] (see also [1, Remark 3.4] for such the differences).

In order to state it, we define

$$\begin{aligned} W(x) &:= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} (x_1 + x_2)/3 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (x_3 + x_4 + x_5 + x_6), \\ S_l^n(x; \alpha, \beta) &:= S_l^n(1, 1, 1) + S_l^n(\alpha, \alpha, 1)x_1 + S_l^n(\beta, \beta, 1)x_2 + S_l^n(\alpha, \alpha, \alpha)x_3 \\ &\quad + S_l^n(\alpha, \alpha, \beta)x_4 + S_l^n(\alpha, \beta, \beta)x_5 + S_l^n(\beta, \beta, \beta)x_6 \end{aligned}$$

for each  $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6$  and  $\alpha, \beta \in [-1, 1)$ . We remark that  $W(x)$  is a symmetric matrix of size 2 and  $S_l^n(x; \alpha, \beta)$  is a symmetric matrix of infinite size indexed by  $\{(i, j) \mid i, j = 0, 1, 2, \dots\}$ .

**Fact 3.1** (Bachoc–Vallentin [1] and Barg–Yu [4]). *Let us fix  $\alpha, \beta \in [-1, 1)$ . Then any finite subset  $X$  of  $S^{n-1}$  with  $I(X) \subset \{\alpha, \beta\}$  satisfies*

$$|X| \leq \max\{1 + (x_1 + x_2)/3 \mid x = (x_1, \dots, x_6) \in \Omega_{\alpha, \beta}^n\}$$

where the subset  $\Omega_{\alpha, \beta}^n$  of  $\mathbb{R}^6$  is defined by

$\Omega_{\alpha, \beta}^n := \{x = (x_1, \dots, x_6) \in \mathbb{R}^6 \mid x \text{ satisfies the following four conditions}\}.$

- (1)  $x_i \geq 0$  for each  $i = 1, \dots, 6$ .
- (2)  $W(x)$  is positive semi-definite.
- (3)  $3 + P_l^n(\alpha)x_1 + P_l^n(\beta)x_2 \geq 0$  for each  $l = 1, 2, \dots$ .
- (4) Any finite principal minor of  $S_l^n(x; \alpha, \beta)$  is positive semi-definite for each  $l = 0, 1, 2, \dots$ .

To prove Theorem 2.1, we use the following “linear version” of Fact 3.1:

**Corollary 3.2.** *In the same setting of Fact 3.1,*

$$|X| \leq \max\{1 + (x_1 + x_2)/3 \mid x = (x_1, \dots, x_6) \in \tilde{\Omega}_{\alpha, \beta}^n\}$$

where the subset  $\tilde{\Omega}_{\alpha, \beta}^n$  of  $\mathbb{R}^6$  is defined by

$\tilde{\Omega}_{\alpha, \beta}^n := \{x = (x_1, \dots, x_6) \in \mathbb{R}^6 \mid x \text{ satisfies the following three conditions}\}.$

- (1)  $x_i \geq 0$  for each  $i = 1, \dots, 6$ .
- (2)  $\det W(x) \geq 0$ .
- (3)  $(S_l^n)_{i,i}(x; \alpha, \beta) \geq 0$  for each  $l, i = 0, 1, 2, \dots$ , where  $(S_l^n)_{i,i}(x; \alpha, \beta)$  is the  $(i, i)$ -entry of the matrix  $S_l^n(x; \alpha, \beta)$ .

By Corollary 3.2, the proof of Theorem 2.1 can be reduced to show the following proposition:

**Proposition 3.3.** *Let  $n \geq 3$  and  $0 < \alpha < 1$ . Then the following holds:*

(1)

$$\max\{1 + (x_1 + x_2)/3 \mid x \in \tilde{\Omega}_{\alpha, -\alpha}^n\} \leq 2 + 2 + (n-2) \frac{(1-\alpha)^3}{\alpha((n-2)\alpha^2 + 6\alpha - 3)}$$

$$\text{if } (1-\alpha)^3(-(n-2)\alpha^2 + 6\alpha + 3) \geq (1+\alpha)^3((n-2)\alpha^2 + 6\alpha - 3) \geq 0.$$

(2)

$$\max\{1 + (x_1 + x_2)/3 \mid x \in \tilde{\Omega}_{\alpha, -\alpha}^n\} \leq 2 + (n-2) \frac{(1+\alpha)^3}{\alpha(-(n-2)\alpha^2 + 6\alpha + 3)}$$

$$\text{if } (1+\alpha)^3((n-2)\alpha^2 + 6\alpha - 3) \geq (1-\alpha)^3(-(n-2)\alpha^2 + 6\alpha + 3) \geq 0.$$

For the proof of Proposition 3.3, we need the next explicit formula of  $(S_3^n)_{1,1}$  which are obtained by direct computations:

**Lemma 3.4.** *For each  $-1 < \alpha < 1$ ,*

$$\begin{aligned} (S_3^n)_{1,1}(1, 1, 1) &= 0 \\ (S_3^n)_{1,1}(\alpha, \alpha, 1) &= \frac{n(n+2)(n+4)(n+6)}{3(n-1)(n+1)(n+3)}\alpha^2(1-\alpha^2)^3 \\ (S_3^n)_{1,1}(\alpha, \alpha, \alpha) &= -\frac{n(n+2)(n+4)(n+6)}{(n-2)(n-1)(n+1)(n+3)}(\alpha-1)^3\alpha^3((n-2)\alpha^2-6\alpha-3) \\ (S_3^n)_{1,1}(\alpha, \alpha, -\alpha) &= -\frac{n(n+2)(n+4)(n+6)}{(n-2)(n-1)(n+1)(n+3)}\alpha^3(\alpha+1)^3((n-2)\alpha^2+6\alpha-3). \end{aligned}$$

*Proof of Proposition 3.3.* Fix  $\alpha$  with  $0 < \alpha < 1$  and take any  $x \in \tilde{\Omega}_{\alpha, -\alpha}^n$ . For simplicity we put  $X = (x_1 + x_2)/3$ ,  $Y = x_3 + x_5$  and  $Z = x_4 + x_6$ . By computing  $\det W(x)$ , we have

$$(1) \quad -X(X-1) + Y + Z \geq 0.$$

Furthermore, we have  $(S_3^n)_{1,1}(x; \alpha, -\alpha) \geq 0$ , and hence, by Lemma 3.4,

$$\begin{aligned} (n-2)\frac{(1-\alpha^2)^3}{\alpha}X - (1-\alpha)^3(-(n-2)\alpha^2+6\alpha+3)Y \\ - (1+\alpha)^3((n-2)\alpha^2+6\alpha-3)Z \geq 0 \end{aligned}$$

Therefore, in the cases where

$$(1-\alpha)^3(-(n-2)\alpha^2+6\alpha+3) \geq (1+\alpha)^3((n-2)\alpha^2+6\alpha-3) \geq 0,$$

we obtain

$$(n-2)\frac{(1-\alpha^2)^3}{\alpha}X - (1+\alpha)^3((n-2)\alpha^2+6\alpha-3)(Y+Z) \geq 0.$$

By combining with (1),

$$(n-2)\frac{(1-\alpha^2)^3}{\alpha}X - (1+\alpha)^3((n-2)\alpha^2+6\alpha-3)X(X-1) \geq 0$$

Thus we have

$$2 + (n-2)\frac{(1-\alpha)^3}{\alpha((n-2)\alpha^2+6\alpha-3)} \geq X + 1 = 1 + (x_1 + x_2)/3.$$

By the similar arguments, in the cases where

$$(1+\alpha)^3((n-2)\alpha^2+6\alpha-3) \geq (1-\alpha)^3(-(n-2)\alpha^2+6\alpha+3) \geq 0,$$

we have

$$2 + (n-2)\frac{(1+\alpha)^3}{\alpha(-(n-2)\alpha^2+6\alpha+3)} \geq X + 1 = 1 + (x_1 + x_2)/3.$$

□

**Remark 3.5.** *Harmonic index 4-designs are defined by using the functional space  $\text{Harm}_4(S^{n-1})$ . Therefore, it seems to be natural to consider  $\text{Harm}_4(S^{n-1})$  in Bachoc–Vallentin’s SDP method. In our proof, the functional space*

$$H_{3,4}^{n-1} \subset \bigoplus_{m=0}^4 H_{m,4}^{n-1} = \text{Harm}_4(S^{n-1})$$

(see [1] for the notation of  $H_{m,l}^{n-1}$ ) plays an important role to show the nonexistence of tight designs of harmonic index 4 since  $(S_3^n)_{1,1}$  comes from  $H_{3,4}^{n-1}$ . We checked that if we consider  $H_{0,4}^{n-1} \oplus H_{1,4}^{n-1} \oplus H_{2,4}^{n-1} \oplus H_{4,4}^{n-1}$  instead of  $H_{3,4}^{n-1}$ , our upper bound can not be obtained for small  $k$ . However, we can not find any conceptional reason of the importance of  $H_{3,4}^{n-1}$ .

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